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Discrete Geometry on 3 Colored Point sets in the Plane

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1 3 colored point sets in the plane

Let R, B and G denote disjoint sets of red points, blue points and green points in the plane, respectively. If no three points of $R \cup B \cup G$ are collinear, we say that R, B and G are *in general position* in the plane. We always assume that given sets of colored points are in general position.

We begin with the following well-known theorem on two colored point sets in the plane. Notice that a *geometric graph* is a graph drawn in the plane whose edges are straight line segments, and every edge of an *alternating matching* joins two points with distinct colors.

Theorem 1 ([3]). *If $|R| = |B|$, then there exists an alternating non-crossing geometric perfect matching on $R \cup B$ (see Figure 1).*

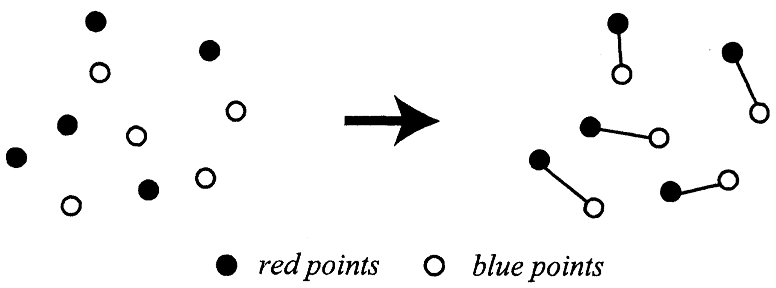

Figure 1: An alternating non-crossing geometric perfect matching on $R \cup B$.

We generalize the above theorem by considering 3 colored point sets. The standard proof of the following theorem is basically similar to that of the above Theorem 1, but more difficult.

Corollary 2 (Kano, Suzuki, Uno [4]). *If $|R \cup B \cup G| = 2n$, $|R| \leq n$, $|B| \leq n$ and $|G| \leq n$, then there exists an alternating non-crossing geometric perfect matching on $R \cup B \cup G$.*

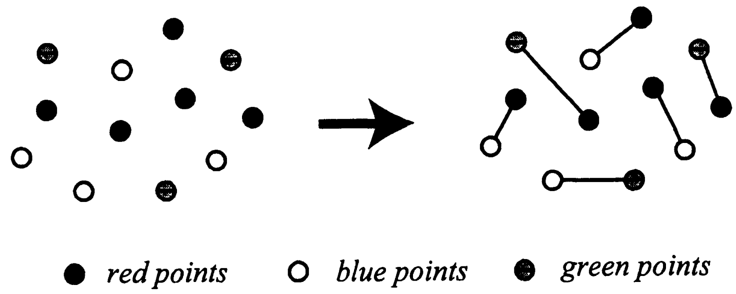

Figure 2: An alternating non-crossing geometric perfect matching on $R \cup B \cup G$.

It is known as the discrete version of Ham-Sandwich theorem that if $|R| = 2m$ and $|B| = 2n$, then there exists a bisector line l such that $|left(l) \cap R| = m$ and $|left(l) \cap B| = n$. It is easy to see that there exist configurations of 3 colored points in the plane such that there exists no line l such that a half-plane determined by l contains the same number of each colored points. Thus the condition in the next theorem is necessary. For a set X of points in the plane, we denote the *convex hull* of X by $conv(X)$.

Theorem 3 (Bereg and Kano [2]). *Assume that $|R| = |B| = |G| = n$, where $n \geq 2$. If all the vertices of $conv(R \cup B \cup G)$ are red, then there exists a line l such that $|right(l) \cap R| = |right(l) \cap B| = |right(l) \cap G| = k$ for some integer $1 \leq k \leq n-1$ (see Figure 3).*

We give one more result on three colored point sets in the plane, and explain a sketch of its proof.

Theorem 4 (Berege and etc. [1]). *Assume that n red points and n blue points and n green points lie on a circle in the plane. Then for every integer $1 \leq k \leq n-1$, there exist two intervals I and J on the circle such that $I \cup J$ contains exactly k red points, k blue ponts and k green points (see Figure 4).*

We give a sketch of its proof.

Lemma 5. *Let $n \geq 2$ be an integer. Then every integer $1 \leq k \leq n-1$ can be obtained from n by applying the following functions f and g some times.*

$$f(x) = \lfloor x/2 \rfloor \qquad \textit{and} \qquad g(x) = n - x$$

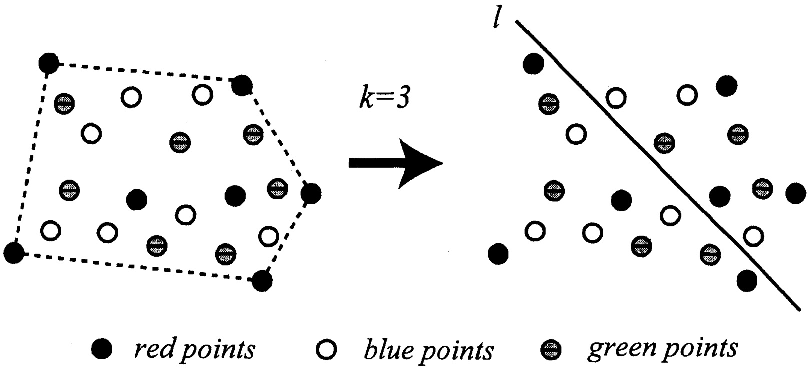

Figure 3: All the vertices of $conv(R \cup B \cup G)$ are red; An line l such that $right(l)$ contains exactly 3 red points, 3 blue points and 3 green points.

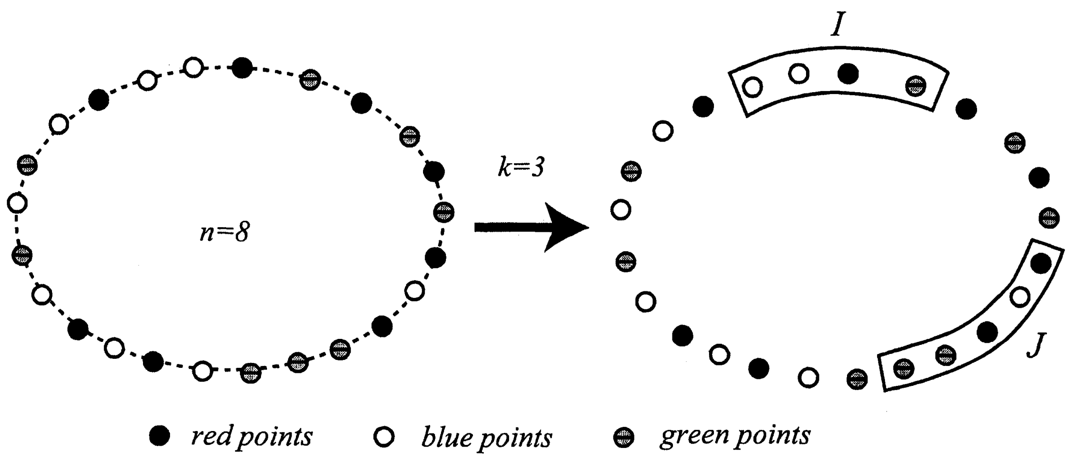

Figure 4: Two disjoint intervals I and J that contains exactly 3 red points, 3 blue points and 3 green points.

We show only one example, whose generalization gives us its proof. Suppose that $n = 30$ and $k = 2$. Then $\lfloor n/2 \rfloor = 15$. We construct the following series of intervals as follows: if an interval $[x, y]$ does not contain 15 and $y < 15$, then make an interval $[2x, 2y + 1]$. If $[x, y]$ does not contain 15 and $15 < x$, then make an interval $[30 - y, 30 - x]$. If an interval $[x, y]$ contains 15, then stop. Then we can obtain $k = 2$ from $\lfloor n/2 \rfloor = 15$ by applying the operations $f(x)$ and $g(x)$ as follows.

$$k = 2 \to [4,5] \to [8,11] \to [16,23] \to [7,14] \to [14,29] \ni 15$$

$$2 \quad \leftarrow \quad 5 \quad \leftarrow \quad 11 \quad \leftarrow \quad 23 \quad \leftarrow \quad 7 \quad \leftarrow \quad 15$$

The next lemma follows immediately from Lemma 5

Lemma 6. *Let $n \geq 2$ be an integer, and let X be a subset of $\{0, 1, 2, \ldots, n\}$. Define two functions f and g as follows:*

$$f(x) = \lfloor x/2 \rfloor \qquad \textit{and} \qquad g(x) = n - x$$

If X has the following properties, then $X = \{0, 1, 2, \ldots, n\}$.

$$n \in X; \ \textit{and if} \quad k \in X, \ \textit{then} \quad g(k) \in X \quad \textit{and} \quad f(k) \in X.$$

Sketch of the proof of Theorem 4. Let us define

$$\begin{aligned} X = \{1 \leq x \leq n \ :\ & \text{there exist two intervals } I \text{ and } J \text{ on the circle} \\ & \text{such that } I \cup J \text{ contains exactly } x \text{ red points,} \\ & x \text{ blue points and } x \text{ green points.}\} \end{aligned}$$

It is easy to see that $n \in X$, and if $k \in X$, then the complement $I \cup J$ on the circle contains exactly $n - k$ red points, $n - k$ blue points and $n - k$ green points, which implies $g(k) = n - k \in X$. Moreover, we can show that if there exist intervals I and J on the circle such that $I \cup J$ contains exactly k red points, k blue points and k green points, then there exist intervals I' and J' in $I \cup J$ such that $I' \cup J'$ contains exactly $\lfloor k/2 \rfloor$ red points $\lfloor k/2 \rfloor$ blue points and $\lfloor k/2 \rfloor$ green points. Hence by Lemma 6, $X = \{0, 1, 2, \ldots, n\}$, which implies that Theorem 4 holds.

References

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[2] S. Bereg and M.kano, Balanced line for a 3-colored point set in the plane, *The Electronic Journal of Combinatorics* **19** (2012) P33.

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[4] M. Kano, K. Suzuki and M. Uno, Geometric graphs on three colored point sets in the plane, (in preparation)